

QUASIDUALIZING MODULES

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ABSTRACT. We introduce and study “quasidualizing” modules. An artinian R -module T is *quasidualizing* if the homothety map $\widehat{R} \rightarrow \text{Hom}_R(T, T)$ is an isomorphism and $\text{Ext}_R^i(T, T) = 0$ for each integer $i > 0$. Quasidualizing modules are associated to semidualizing modules via Matlis duality. We investigate the associations via Matlis duality between subclasses of the Auslander class and Bass class and subclasses of derived T -reflexive modules.

INTRODUCTION

Let R be a commutative local noetherian ring with maximal ideal \mathfrak{m} and residue field $k = R/\mathfrak{m}$. The \mathfrak{m} -adic completion of R is denoted \widehat{R} , the injective hull of k is $E = E_R(k)$, and the Matlis duality functor is $(-)^{\vee} = \text{Hom}_R(-, E)$.

The motivation for this work comes from the study of semidualizing modules. Semidualizing modules were first introduced by Vasconcelos [8]. A finitely generated R -module C is *semidualizing* if the homothety map $R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism and $\text{Ext}_R^i(C, C) = 0$ for each integer $i > 0$. For example, R is always a semidualizing R -module. Therefore duality with respect to R is a special case of duality with respect to a semidualizing module, as is duality with respect to a dualizing R -module when R has one. On the other hand, Matlis duality is not covered in this way. The goal of this paper is to remedy this by introducing and studying the “quasidualizing” modules: An artinian R -module T is *quasidualizing* if the homothety map $\widehat{R} \rightarrow \text{Hom}_R(T, T)$ is an isomorphism and $\text{Ext}_R^i(T, T) = 0$ for each integer $i > 0$; see Definition 1.14. For example, E is always a quasidualizing module.

This paper is concerned with the properties of quasidualizing modules and how they compare with the properties of semidualizing modules. For instance, the next result gives a direct link between quasidualizing modules and semidualizing modules via Matlis duality; see Theorem 3.1.

Theorem A. *If R is complete, then the set of isomorphism classes of semidualizing R -modules is in bijection with the set of isomorphism classes of quasidualizing R -modules by Matlis duality.*

Following the literature on semidualizing modules, we use quasidualizing modules to define other classes of modules. For instance, given an R -module M , we consider the class $\mathcal{G}_M^{\text{full}}(R)$ of “derived M -reflexive R -modules” and their subclasses $\mathcal{G}_M^{\text{noeth}}(R)$ and $\mathcal{G}_M^{\text{artin}}(R)$ of noetherian modules and artinian modules respectively. We also consider subclasses of the Auslander class $\mathcal{A}_M(R)$ and the Bass class $\mathcal{B}_M(R)$. See

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Section 1 for definitions. Some relations between these classes are listed in the next result which is proved in Section 3.

Theorem B. *Assume R is complete, and let T be a quasidualizing R -module. Then we have the following inverse equivalences and equalities*

$$\begin{aligned}
\text{(i)} \quad & \mathcal{B}_{T^\vee}^{\text{noeth}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_T^{\text{artin}}(R) = \mathcal{A}_{T^\vee}^{\text{artin}}(R) ; \\
\text{(ii)} \quad & \mathcal{B}_{T^\vee}^{\text{artin}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_T^{\text{noeth}}(R) = \mathcal{A}_{T^\vee}^{\text{noeth}}(R) ; \\
\text{(iii)} \quad & \mathcal{B}_T^{\text{artin}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_{T^\vee}^{\text{noeth}}(R) = \mathcal{A}_T^{\text{noeth}}(R) ; \text{ and} \\
\text{(iv)} \quad & \mathcal{B}_T^{\text{noeth}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_{T^\vee}^{\text{artin}}(R) = \mathcal{A}_T^{\text{artin}}(R) .
\end{aligned}$$

As a consequence of the previous result, we conclude that the classes $\mathcal{G}_{T^\vee}^{\text{noeth}}(R)$ and $\mathcal{G}_T^{\text{artin}}(R)$ are substantially different. For instance, as we observe next $\mathcal{G}_T^{\text{artin}}(R)$ satisfies the two-of-three condition, while the class $\mathcal{G}_{T^\vee}^{\text{noeth}}(R)$ does not; see Theorem 3.13.

Theorem C. *Assume that R is complete, and let T be a quasidualizing R -module. Then $\mathcal{G}_T^{\text{artin}}(R)$ satisfies the two-of-three condition, that is, given an exact sequence of R -module homomorphisms $0 \rightarrow L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow 0$ if any two of the modules are in $\mathcal{G}_T^{\text{artin}}(R)$, then so is the third.*

In Section 1 we provide some definitions and background material. Section 2 describes properties related to quasidualizing modules, and Section 3 describes the relations between the different classes of modules using Matlis duality as well as Theorem C.

1. BACKGROUND MATERIAL

Definition 1.1. We say that an R -module L is *Matlis reflexive* if the natural biduality map $\delta_L^E : L \rightarrow L^{\vee\vee}$, given by $l \mapsto [\phi \mapsto \phi(l)]$.

Fact 1.2. Let L be an R -module. The natural biduality map δ_L is injective; see [6, Theorem 18.6(i)]. If L is Matlis reflexive, then L^\vee is Matlis reflexive.

Fact 1.3. Assume R is complete and let L be an R -module. If L is artinian, then L^\vee is noetherian. If L is noetherian, then L^\vee is artinian. Since R is complete, both artinian modules and noetherian modules are Matlis reflexive; see [6, Theorem 18.6(v)].

Lemma 1.4. *Let L and L' be R -modules such that L is Matlis reflexive. Then for all $i \geq 0$ we have the isomorphisms*

$$\text{Ext}_R^i(L', L) \cong \text{Ext}_R^i(L^\vee, L'^\vee) \text{ and } \text{Ext}_R^i(L', L^\vee) \cong \text{Ext}_R^i(L, L'^\vee).$$

Proof. For the first isomorphism, since L is Matlis reflexive, by definition the map

$$\text{Ext}_R^i(L', \delta_L) : \text{Ext}_R^i(L', L) \rightarrow \text{Ext}_R^i(L', \text{Hom}_R(L^\vee, E))$$

is an isomorphism. A manifestation of Hom-tensor adjointness yields the following isomorphisms

$$\mathrm{Ext}_R^i(L', \mathrm{Hom}_R(L^\vee, E)) \xrightarrow{\cong} \mathrm{Ext}_R^i(L' \otimes_R L^\vee, E) \xrightarrow{\cong} \mathrm{Ext}_R^i(L^\vee, L'^\vee).$$

The composition of these maps provides us with the isomorphism $\mathrm{Ext}_R^i(L', L) \cong \mathrm{Ext}_R^i(L^\vee, L'^\vee)$.

For the second isomorphism, the fact that L is Matlis reflexive explains the second step in the following sequence $\mathrm{Ext}_R^i(L', L^\vee) \cong \mathrm{Ext}_R^i(L^{\vee\vee}, L'^\vee) \cong \mathrm{Ext}_R^i(L, L'^\vee)$. The first step follows from the first isomorphism since L^\vee is Matlis reflexive. \square

Fact 1.5. Assume R is complete and let A and A' be artinian R -modules. Then $\mathrm{Hom}_R(A, A')$ is noetherian. This can be deduced using [5, Theorem 2.11].

Fact 1.6. Let L be an R -module. Then L is artinian over R if and only if it is artinian over \widehat{R} . See [5, Lemma 1.14].

Lemma 1.7. Assume R is artinian and let L be an R -module. Then the following are equivalent

- (i) L is noetherian over R ;
- (ii) L is finitely generated over R ; and
- (iii) L is artinian.

Proof. The equivalence (i) \iff (ii) is standard; see [1, Propositions 6.2 and 6.5].

For the implication (ii) \implies (iii), assume that L is finitely generated over R . Then there exists an $n \in \mathbb{N}$ and a surjective map $R^n \xrightarrow{\phi} L$ so that we have $L \cong \mathrm{Im}(\phi) \cong R^n / \mathrm{Ker}(\phi)$. Since R is artinian, R^n is artinian. Thus L is artinian because the quotient of an artinian module is artinian; see [1, Proposition 6.3].

For the implication (iii) \implies (i), assume that L is artinian. Then there exists an $n \in \mathbb{N}$ such that $L \hookrightarrow E^n$; see [2, Theorem 3.4.3]. Since R is artinian, we have $R^\vee \cong E$ is noetherian over \widehat{R} by Fact 1.3, where the isomorphism follows from [6, Theorem 18.6 (iv)]. Hence we have that E^n is noetherian over $\widehat{R} = R$ since R is artinian. Since any submodule of a noetherian module is noetherian, we conclude that L is noetherian over R ; see [1, Proposition 6.3]. \square

Lemma 1.8. Assume R is complete and let A be an artinian R -module. Then there exists an injective resolution I of A such that for each $i \geq 0$ we have $I_i \cong E^{b_i}$ for some $b_i \in \mathbb{N}$. Furthermore, I^\vee is a free resolution of A^\vee .

Proof. Since A is artinian, we have the map $A \hookrightarrow E^{b_0}$ for some $b_0 \geq 1$; see [2, Theorem 3.4.3]. Because the finite direct sum of artinian modules is artinian, E^{b_0} is artinian and we have $E^{b_0}/A \hookrightarrow E^{b_1}$ for some $b_1 \geq 0$. Recursively we can construct an injective resolution of A such that for each $i \geq 0$ we have $I_i \cong E^{b_i}$ for some $b_i \in \mathbb{N}$.

Next we show that I^\vee is a free resolution of A^\vee . The fact that $I_i \cong E^{b_i}$ explains the first step in the following sequence

$$I_i^\vee = \mathrm{Hom}_R(I_i, E) \cong \mathrm{Hom}_R(E^{b_i}, E) \cong \mathrm{Hom}_R(E, E)^{b_i} \cong \widehat{R}^{b_i} \cong R^{b_i}.$$

The second step is standard. The third step is from [6, Theorem 18.6(iv)], and the last step follows from the assumption that R is complete. The desired conclusion follows from the fact that $(-)^{\vee}$ is exact. \square

Definition 1.9. Let L, L' and L'' be R -modules. The *Hom-evaluation* morphism

$$\theta_{LL'L''} : L \otimes_R \text{Hom}_R(L', L'') \rightarrow \text{Hom}_R(\text{Hom}_R(L, L'), L'')$$

is given by $a \otimes \phi \mapsto [\beta \mapsto \phi(\beta(a))]$.

Fact 1.10. The Hom-evaluation morphism $\theta_{LL'L''}$ is an isomorphism if the modules satisfy one of the following conditions:

- (a) L is finitely generated and L'' is injective; or
- (b) L is finitely generated and projective.

See [4, Lemma 1.6] and [7, Lemma 3.55].

Definition 1.11. An R -module C is *semidualizing* if it satisfies the following

- (i) C is finitely generated;
- (ii) the homothety morphism $\chi_C^R : R \rightarrow \text{Hom}_R(C, C)$, defined by $r \mapsto [c \mapsto rc]$, is an isomorphism; and
- (iii) one has $\text{Ext}_R^i(C, C) = 0$ for all $i > 0$.

Remark 1.12. Let $\mathfrak{S}_0(R)$ denote the set of isomorphism classes of semidualizing R -modules.

Example 1.13. The ring R is always semidualizing.

Definition 1.14. An R -module T is *quasidualizing* if it satisfies the following

- (i) T is artinian;
- (ii) the homothety morphism $\chi_T^{\widehat{R}} : \widehat{R} \rightarrow \text{Hom}_R(T, T)$, defined by $r \mapsto [t \mapsto rt]$, is an isomorphism; and
- (iii) one has $\text{Ext}_R^i(T, T) = 0$ for all $i > 0$.

Remark 1.15. The homothety morphism $\chi_T^{\widehat{R}}$ is well defined since T is artinian implying by Fact 1.6 that T is an \widehat{R} -module.

Remark 1.16. Let $\mathfrak{Q}_0(R)$ denote the set of isomorphism classes of quasidualizing modules.

Example 1.17. The injective hull of the residue field E is always quasidualizing. See [2, Theorem 3.4.1] and [6, Theorem 18.6(iv)] for conditions (i) and (ii) of Definition 1.14. Since E is injective by definition, we have $\text{Ext}_R^i(E, E) = 0$ for all $i > 0$ satisfying the last condition.

Definition 1.18. Let M be an R -module. Then an R -module L is *derived M -reflexive* if

- (i) the natural biduality map $\delta_L^M : L \rightarrow \text{Hom}_R(\text{Hom}_R(L, M), M)$ defined by $l \mapsto [\phi \mapsto \phi(l)]$ is an isomorphism; and
- (ii) one has $\text{Ext}_R^i(L, M) = 0 = \text{Ext}_R^i(\text{Hom}_R(L, M), M)$ for all $i > 0$.

We write $\mathcal{G}_M^{\text{full}}(R)$ to denote the class of all derived M -reflexive R -modules, $\mathcal{G}_M^{\text{mr}}(R)$ to denote the class of all Matlis reflexive derived M -reflexive R -modules, $\mathcal{G}_M^{\text{artin}}(R)$ to denote the class of all artinian derived M -reflexive R -modules, and $\mathcal{G}_M^{\text{noeth}}(R)$ to denote the class of all noetherian derived M -reflexive R -modules.

Remark 1.19. When $M = C$ is a semidualizing R -module, the class $\mathcal{G}_M^{\text{noeth}}(R)$ is the class of *totally C -reflexive* R -modules, sometimes denoted $\mathcal{G}_C(R)$.

Definition 1.20. Let L and L' be R -modules. We say that L is in the *Bass class* $\mathcal{B}_{L'}(R)$ with respect to L' if it satisfies the following:

- (i) the natural evaluation homomorphism $\xi_L^{L'} : \text{Hom}_R(L', L) \otimes_R L' \rightarrow L$, defined by $\phi \otimes l \mapsto \phi(l)$, is an isomorphism; and
- (ii) one has $\text{Ext}_R^i(L', L) = 0 = \text{Tor}_i^R(L', \text{Hom}_R(L', L))$ for all $i > 0$.

We write $\mathcal{B}_{L'}^{\text{mr}}(R)$ to denote the class of all Matlis reflexive R -modules in the Bass class with respect to L' . We write $\mathcal{B}_{L'}^{\text{artin}}(R)$ to denote the class of all artinian R -modules in the Bass class with respect to L' , and $\mathcal{B}_{L'}^{\text{noeth}}(R)$ to denote the class of all noetherian R -modules in the Bass class with respect to L' .

Definition 1.21. Let L and L' be R -modules. We say that L is in the *Auslander class* $\mathcal{A}_{L'}(R)$ with respect to L' if it satisfies the following

- (i) the natural homomorphism $\gamma_L^{L'} : L \rightarrow \text{Hom}_R(L', L' \otimes_R L)$, which is defined by $l \mapsto [l' \mapsto l' \otimes l]$, is an isomorphism; and
- (ii) one has $\text{Tor}_i^R(L', L) = 0 = \text{Ext}_R^i(L', L' \otimes_R L)$ for all $i > 0$.

We write $\mathcal{A}_{L'}^{\text{mr}}(R)$ to denote the class of all Matlis reflexive R -modules in the Auslander class with respect to L' . We write $\mathcal{A}_{L'}^{\text{artin}}(R)$ to denote the class of all artinian R -modules in the Auslander class with respect to L' , and $\mathcal{A}_{L'}^{\text{noeth}}(R)$ to denote the class of all noetherian R -modules in the Auslander class with respect to L' .

2. QUASIDUALIZING MODULES

We begin with a few preliminary results pertaining to quasidualizing modules.

Proposition 2.1. *Let T be an R -module. Then T is a quasidualizing R -module if and only if T is a quasidualizing \widehat{R} -module.*

Proof. We need to check the equivalence of three conditions. For the first condition, T is an artinian R -module if and only if T is an artinian \widehat{R} -module by Fact 1.6. For the rest of the proof we assume without loss of generality that T is artinian.

For the second condition, we have the equality $\text{Hom}_R(T, T) = \text{Hom}_{\widehat{R}}(T, T)$ from the fact that T is \mathfrak{m} -torsion and [5, Lemma 1.5(a)]. This explains the equality in the following commutative diagram.

$$\begin{array}{ccc} \widehat{R} & \xrightarrow{\chi_T^{\widehat{R}}} & \text{Hom}_R(T, T) \\ \cong \downarrow & & \downarrow = \\ \widehat{\widehat{R}} & \xrightarrow{\chi_T^{\widehat{\widehat{R}}}} & \text{Hom}_{\widehat{R}}(T, T) \end{array}$$

Since $\widehat{R} \cong \widehat{\widehat{R}}$, we have $\chi_T^{\widehat{R}}$ is an isomorphism if and only if $\chi_T^{\widehat{\widehat{R}}}$ is an isomorphism.

For the last condition, Lemma 1.8 implies that there exists an injective resolution I of T such that for each $i \geq 0$ we have $I_i \cong E^{b_i}$ for some $b_i \in \mathbb{N}$. For all $i \geq 0$, the modules T and I_i are artinian and hence \mathfrak{m} -torsion. By [5, Lemma 1.5(a)], we have the equality $\text{Hom}_{\widehat{R}}(T, I_i) = \text{Hom}_R(T, I_i)$ and I is an injective resolution of T over \widehat{R} . This explains the first and second steps in the next display:

$$\text{Ext}_{\widehat{R}}^i(T, T) \cong \text{H}_{-i}(\text{Hom}_{\widehat{R}}(T, I_i)) \cong \text{H}_{-i}(\text{Hom}_R(T, I_i)) \cong \text{Ext}_R^i(T, T).$$

The third step is by definition. Thus we have $\text{Ext}_{\widehat{R}}^i(T, T) = 0$ for all $i > 0$ if and only if $\text{Ext}_R^i(T, T) = 0$ for all $i > 0$. \square

Proposition 2.2. *The following conditions are equivalent*

- (i) E is a semidualizing R -module;
- (ii) R is a quasidualizing R -module;
- (iii) E is a noetherian R -module;
- (iv) R is an artinian ring;
- (v) $\mathfrak{Q}_0(R) = \mathfrak{S}_0(R)$; and
- (vi) $\mathfrak{Q}_0(R) \cap \mathfrak{S}_0(R) \neq 0$.

Proof. (iii) \iff (iv) By [6, Theorem 18.6 (ii)] we have $\text{len}_R(R) = \text{len}_R(R^\vee) = \text{len}_R(E)$, where $\text{len}_R(L)$ denotes the length of an R -module L . Since R is noetherian by assumption, we have R is artinian if and only if R has finite length if and only if $R^\vee = E$ has finite length (by the equalities above), if and only if E is noetherian over R (since E is artinian; see [2, Theorem 3.4.1]). That is, R is artinian if and only if E is noetherian over R .

(i) \implies (iii) If E is a semidualizing R -module, then E is noetherian over R by definition.

(iv) \implies (i) Assume that R is artinian. Then E is finitely generated by the equivalence (iii) \iff (iv). We have $R \cong \widehat{R}$ since R is artinian, and $\widehat{R} \cong \text{Hom}_R(E, E)$ by [6, Theorem 18.6 (iv)] explaining the unspecified isomorphisms in the following commutative diagram.

$$\begin{array}{ccc} R & \xrightarrow{\chi_E^R} & \text{Hom}_R(E, E) \\ \cong \downarrow & \nearrow \cong & \\ \widehat{R} & & \end{array}$$

Hence we conclude that the homothety morphism χ_E^R is an isomorphism. Since E is injective, we have that $\text{Ext}_R^i(E, E) = 0$ for all $i > 0$. Thus E is a semidualizing R -module.

(iv) \implies (v) Assume that R is artinian, and let L be an R -module. We show that L is a semidualizing module if and only if L is a quasidualizing module. We need to check the equivalence of three conditions. For the first condition, L is finitely generated if and only if L is artinian by Lemma 1.7. For the second condition, the fact that R is artinian implies that $\widehat{R} \cong R$. This explains the unlabeled isomorphism in the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\cong} & \widehat{R} \\ \chi_L^R \downarrow & \nearrow \chi_L^{\widehat{R}} & \\ \text{Hom}_R(L, L) & & \end{array}$$

Thus the map χ_L^R is an isomorphism if and only if the map $\chi_L^{\widehat{R}}$ is an isomorphism. The Ext vanishing conditions are equivalent by definition.

For the implication (v) \implies (ii), assume that $\mathfrak{Q}_0(R) = \mathfrak{S}_0(R)$. The R -module R is always semidualizing. Then by assumption it is also a quasidualizing R -module.

The implication (ii) \implies (iv) is evident since R is an artinian ring if and only if it is an artinian R -module. For the implication (ii) \implies (vi), if R is a quasidualizing R -module, then the intersection $\mathfrak{Q}_0(R) \cap \mathfrak{S}_0(R)$ is nonempty since R is also a semidualizing R -module.

For the implication (vi) \implies (ii), assume that the intersection $\mathfrak{Q}_0(R) \cap \mathfrak{S}_0(R)$ is nonempty. Let $L \in \mathfrak{Q}_0(R) \cap \mathfrak{S}_0(R)$. Then L is artinian and noetherian, so it has finite length. Since L is artinian, it is \mathfrak{m} -torsion and by [5, Fact 1.2(b)] we have $\text{Supp}_R(L) \subseteq \{\mathfrak{m}\}$. Since L is a semidualizing R -module, the map $R \rightarrow \text{Hom}_R(L, L)$ is an isomorphism so we have $\text{Ann}_R(L) \subseteq \text{Ann}_R(R) = 0$. This explains the second step in the following sequence

$$\text{Supp}_R(L) = V(\text{Ann}_R(L)) = V(0) = \text{Spec}(R).$$

Thus $\text{Spec}(R) = \text{Supp}_R(L) \subseteq \{\mathfrak{m}\} \subseteq \text{Spec}(R)$ and we conclude that $\text{Spec}(R) = \{\mathfrak{m}\}$. Thus [1, Theorem 8.5] implies that R is artinian. \square

3. CLASSES OF MODULES AND MATLIS DUALITY

This section explores the connections between the class of quasidualizing R -modules and the class of semidualizing R -modules as well as connections between different subclasses of $\mathcal{A}_M(R)$, $\mathcal{B}_M(R)$, and $\mathcal{G}_M^{\text{full}}(R)$. The instrument used to detect these connections is Matlis Duality.

Theorem 3.1. *Assume that R is complete. Then the maps $\mathfrak{S}_0(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathfrak{Q}_0(R)$ are inverse bijections.*

Proof. Let $C \in \mathfrak{S}_0(R)$. We show that $C^\vee \in \mathfrak{Q}_0(R)$. Fact 1.3 implies that C^\vee is artinian. In the following commutative diagram, the unspecified isomorphisms are from Hom-tensor adjointness and the commutativity of tensor product

$$\begin{array}{ccc} R & \xrightarrow{\chi_{C^\vee}^R} & \text{Hom}_R(C^\vee, \text{Hom}_R(C, E)) \\ \downarrow \chi_C^R & & \downarrow \cong \\ \text{Hom}_R(C, C) & & \text{Hom}_R(C^\vee \otimes_R C, E) \\ \cong \downarrow \text{Hom}_R(C, \delta_C^E) & & \downarrow \cong \\ \text{Hom}_R(C, \text{Hom}_R(C^\vee, E)) & \xrightarrow{\cong} & \text{Hom}_R(C \otimes_R C^\vee, E). \end{array}$$

Since $C \in \mathfrak{S}_0(R)$, it follows that χ_C^R is an isomorphism. Fact 1.3 implies that the map δ_C^E , and by extension the map $\text{Hom}_R(C, \delta_C^E)$, is an isomorphism. Hence we conclude from the diagram that $\chi_{C^\vee}^R$ is an isomorphism.

For the last condition, Lemma 1.4 explains the first step in the following sequence

$$\text{Ext}_R^i(C^\vee, C^\vee) \cong \text{Ext}_R^i(C, C) = 0.$$

The second step follows from the fact that C is a semidualizing module. Thus C^\vee is a quasidualizing module.

A similar argument shows that given a quasidualizing R -module T , the module T^\vee is semidualizing. Fact 1.3 implies that $C \cong C^{\vee\vee}$ and $T \cong T^{\vee\vee}$, so that the given maps $\mathfrak{S}_0(R) \xrightarrow{(-)^\vee} \mathfrak{Q}_0(R)$ and $\mathfrak{Q}_0(R) \xrightarrow{(-)^\vee} \mathfrak{S}_0(R)$ are inverse equivalences. \square

Example 3.2. Assume that R is Cohen-Macaulay and complete and admits a dualizing module D . The fact that D is dualizing means that D is semidualizing and has finite injective dimension. Therefore, by Theorem 3.1, we conclude that D^\vee is quasidualizing.

Proposition 3.3. *Assume that R is complete and let T be a quasidualizing R -module. Then the maps $\mathcal{B}_{T^\vee}^{\text{mr}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_T^{\text{mr}}(R)$ are inverse bijections.*

Proof. Let M be a Matlis reflexive R -module. We show that if $M \in \mathcal{B}_{T^\vee}^{\text{mr}}(R)$ then $M^\vee \in \mathcal{G}_T^{\text{mr}}(R)$. Fact 1.2 implies that M^\vee is Matlis reflexive. There are three remaining conditions to check.

First we show that $\text{Ext}_R^i(M^\vee, T) = 0$ for all $i > 0$. Since T is artinian and R is complete, Fact 1.3 implies that T is Matlis reflexive, so we have

$$(3.3.1) \quad \text{Ext}_R^i(M^\vee, T) \cong \text{Ext}_R^i(T^\vee, M).$$

by Lemma 1.4. We have $\text{Ext}_R^i(T^\vee, M) = 0$ for all $i > 0$ since $M \in \mathcal{B}_{T^\vee}^{\text{mr}}(R)$. Thus we conclude $\text{Ext}_R^i(M^\vee, T) = 0$ for all $i > 0$.

Next we show that the map $\delta_{M^\vee}^T$ is an isomorphism. The fact that $M \in \mathcal{B}_{T^\vee}^{\text{mr}}(R)$ implies the map $\xi_M^{T^\vee}$ is an isomorphism. Therefore the map $\text{Hom}_R(\xi_M^{T^\vee}, E)$ in the following commutative diagram is an isomorphism

$$\begin{array}{ccc} M^\vee & \xrightarrow[\cong]{\text{Hom}_R(\xi_M^{T^\vee}, E)} & \text{Hom}_R(\text{Hom}_R(T^\vee, M) \otimes_R T^\vee, E) \\ \delta_{M^\vee}^T \downarrow & & \downarrow \cong \\ \text{Hom}_R(\text{Hom}_R(M^\vee, T), T) & \xrightarrow[\cong]{} & \text{Hom}_R(\text{Hom}_R(T^\vee, M), T). \end{array}$$

The unspecified isomorphisms are from Hom-tensor adjointness and the isomorphism (3.3.1). Hence we conclude from the diagram that $\delta_{M^\vee}^T$ is an isomorphism.

For the last condition, let I be an injective resolution of T such that for each $i \geq 0$ we have $I_i \cong E^{b_i}$ for some $b_i \in \mathbb{N}$. Lemma 1.8 implies that I^\vee is a free resolution of T^\vee . This explains steps (2) and (6) in the following sequence

$$\begin{aligned} \text{Ext}_R^i(\text{Hom}_R(M^\vee, T), T) &\stackrel{(1)}{\cong} \text{Ext}_R^i(\text{Hom}_R(T^\vee, M), T) \\ &\stackrel{(2)}{\cong} \text{H}_{-i}(\text{Hom}_R(\text{Hom}_R(T^\vee, M), I)) \\ &\stackrel{(3)}{\cong} \text{H}_{-i}(\text{Hom}_R(\text{Hom}_R(T^\vee, M), I^{\vee\vee})) \\ &\stackrel{(4)}{\cong} \text{H}_{-i}(\text{Hom}_R(\text{Hom}_R(T^\vee, M) \otimes_R I^\vee, E)) \\ &\stackrel{(5)}{\cong} \text{Hom}_R(\text{H}_i(I^\vee \otimes_R \text{Hom}_R(T^\vee, M)), E) \\ &\stackrel{(6)}{\cong} \text{Hom}_R(\text{Tor}_i^R(T^\vee, \text{Hom}_R(T^\vee, M)), E). \end{aligned}$$

Step (1) follows from the isomorphism (3.3.1). Step (3) follows from the fact that any finite direct sum of artinian modules is artinian; thus I_j is artinian for all j and we can apply Fact 1.3. Step (4) follows from Hom-tensor adjointness, and step (5) follows from the fact that E is injective and homology commutes with exact functors. Since $M \in \mathcal{B}_{T^\vee}^{\text{mr}}(R)$, we have $\text{Tor}_i^R(M, \text{Hom}_R(T^\vee, M)) = 0$ for all $i > 0$. Hence we conclude that

$$\text{Ext}_R^i(\text{Hom}_R(M^\vee, T), T) \cong \text{Hom}_R(\text{Tor}_i^R(M, \text{Hom}_R(T^\vee, M)), E) = 0$$

for all $i > 0$.

Given an R -module $M' \in \mathcal{G}_T^{\text{mr}}(R)$, the argument to show that $M'^\vee \in \mathcal{B}_{T^\vee}^{\text{mr}}(R)$ is similar. Since M and M' are Matlis reflexive, that is $M \cong M^{\vee\vee}$ and $M' \cong M'^{\vee\vee}$, we conclude that the maps $\mathcal{B}_{T^\vee}^{\text{mr}}(R) \xrightarrow{(-)^\vee} \mathcal{G}_T^{\text{mr}}(R)$ and $\mathcal{G}_T^{\text{mr}}(R) \xrightarrow{(-)^\vee} \mathcal{B}_{T^\vee}^{\text{mr}}(R)$ are inverse equivalences. \square

Corollary 3.4. *Assume that R is complete and let T be a quasidualizing R -module. Then the following maps are inverse bijections*

$$\mathcal{B}_{T^\vee}^{\text{noeth}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_T^{\text{artin}}(R) \quad \text{and} \quad \mathcal{B}_{T^\vee}^{\text{artin}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_T^{\text{noeth}}(R).$$

Proof. Fact 1.3 implies that if N is a noetherian R -module, then N^\vee is an artinian R -module and $N \cong N^{\vee\vee}$. Furthermore, if A is an artinian R -module, then A^\vee is a noetherian R -module and $A \cong A^{\vee\vee}$. Together with Proposition 3.3, this implies

that the maps $\mathcal{B}_{T^\vee}^{\text{noeth}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_T^{\text{artin}}(R)$ are inverse bijections. The proof for

$\mathcal{B}_{T^\vee}^{\text{artin}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_T^{\text{noeth}}(R)$ is similar. \square

Proposition 3.5. *Assume that R is complete and let T be a quasidualizing R -module. Then the maps $\mathcal{B}_T^{\text{mr}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_{T^\vee}^{\text{mr}}(R)$ are inverse bijections.*

Proof. Let M be a Matlis reflexive R -module. We show that if $M \in \mathcal{G}_{T^\vee}^{\text{mr}}(R)$, then $M^\vee \in \mathcal{B}_T^{\text{mr}}(R)$. First we show that the map $\xi_{M^\vee}^T$ is an isomorphism. The fact that M is Matlis reflexive implies that the map δ_M^E in the following commutative diagram is an isomorphism

$$\begin{array}{ccc} M & \xrightarrow[\cong]{\delta_M^E} & M^{\vee\vee} \\ \downarrow \delta_M^{T^\vee} & & \downarrow (\xi_{M^\vee}^T)^\vee \\ \text{Hom}_R(\text{Hom}_R(M, T^\vee), T^\vee) & & \text{Hom}_R(\text{Hom}_R(T, M^\vee) \otimes_R T, E) \\ \downarrow \cong & \nearrow \cong & \\ \text{Hom}_R(\text{Hom}_R(T, M^\vee), T^\vee) & & \end{array}$$

The unspecified isomorphisms are from Hom-tensor adjointness and Lemma 1.4. Since $M \in \mathcal{G}_{T^\vee}^{\text{mr}}(R)$, we have that the map $\delta_M^{T^\vee}$ is an isomorphism. Hence $(\xi_{M^\vee}^T)^\vee$ is an isomorphism. Since E is faithfully injective, this implies that $\xi_{M^\vee}^T$ is an isomorphism.

Next we show that $\text{Ext}_R^i(T, M^\vee) = 0$ for all $i > 0$. Since M is Matlis reflexive, Lemma 1.4 explains the first step in the following sequence $\text{Ext}_R^i(T, M^\vee) \cong \text{Ext}_R^i(M, T^\vee) = 0$. The second step follows from the fact that $M \in \mathcal{G}_{T^\vee}^{\text{mr}}(R)$.

Lastly, we show that $\mathrm{Tor}_i^R(T, \mathrm{Hom}_R(T, M^\vee)) = 0$ for all $i > 0$. The commutativity of tensor product explains the first step in the following sequence

$$\begin{aligned} \mathrm{Tor}_i^R(T, \mathrm{Hom}_R(T, M^\vee))^\vee &\cong \mathrm{Tor}_i^R(\mathrm{Hom}_R(T, M^\vee), T)^\vee \\ &\cong \mathrm{Ext}_R^i(\mathrm{Hom}_R(T, M^\vee), T^\vee) \\ &\cong \mathrm{Ext}_R^i(\mathrm{Hom}_R(M, T^\vee), T^\vee) \\ &= 0. \end{aligned}$$

The second step follows from [5, Remark 1.9] and the third step follows from Lemma 1.4. The last step follows from the fact that $M \in \mathcal{G}_{T^\vee}^{\mathrm{mr}}(R)$.

Given an R -module $M' \in \mathcal{B}_T^{\mathrm{mr}}(R)$, the argument to show that $M'^\vee \in \mathcal{G}_T^{\mathrm{mr}}(R)$ is similar but easier. Since M and M' are Matlis reflexive, we conclude that the maps $\mathcal{B}_T^{\mathrm{mr}}(R) \xrightarrow{(-)^\vee} \mathcal{G}_{T^\vee}^{\mathrm{mr}}(R)$ and $\mathcal{G}_T^{\mathrm{mr}}(R) \xrightarrow{(-)^\vee} \mathcal{B}_T^{\mathrm{mr}}(R)$ are inverse equivalences. \square

Corollary 3.6. *Assume that R is complete and let T be a quasidualizing R -module. Then the following maps are inverse bijections*

$$\mathcal{B}_T^{\mathrm{noeth}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_{T^\vee}^{\mathrm{artin}}(R) \quad \text{and} \quad \mathcal{B}_T^{\mathrm{artin}}(R) \xrightleftharpoons[(-)^\vee]{(-)^\vee} \mathcal{G}_{T^\vee}^{\mathrm{noeth}}(R).$$

The next proposition establishes the relationship between a subclass of the Auslander class and a subclass of the derived reflexive modules.

Proposition 3.7. *If R is complete and T is a quasidualizing R -module, then*

$$\mathcal{G}_{T^\vee}^{\mathrm{mr}}(R) = \mathcal{A}_T^{\mathrm{mr}}(R).$$

Proof. Let M be a Matlis reflexive R -module. We show that M satisfies the defining conditions of $\mathcal{G}_{T^\vee}^{\mathrm{mr}}(R)$ if and only if M satisfies the defining conditions of $\mathcal{A}_T^{\mathrm{mr}}(R)$. For the isomorphisms, consider the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\delta_M^{T^\vee}} & \mathrm{Hom}_R(\mathrm{Hom}_R(M, T^\vee), T^\vee) \\ \downarrow \gamma_M^T & & \downarrow \cong \\ \mathrm{Hom}_R(T, T \otimes_R M) & & \mathrm{Hom}_R(\mathrm{Hom}_R(M, T^\vee) \otimes_R T, E) \\ \cong \downarrow \mathrm{Hom}_R(T, \delta_{T \otimes_R M}^E) & & \downarrow \cong \\ \mathrm{Hom}_R(T, \mathrm{Hom}_R((T \otimes_R M)^\vee, E)) & \xrightarrow{\cong} & \mathrm{Hom}_R(T, \mathrm{Hom}_R(\mathrm{Hom}_R(M, T^\vee), E)). \end{array}$$

The unspecified isomorphisms are Hom-tensor adjointness. The module $T \otimes_R M$ is artinian by [5, Lemma 1.19 and Theorem 3.1]. Fact 1.3 implies that the map $\delta_{T \otimes_R M}^E$, and hence the map $\mathrm{Hom}_R(T, \delta_{T \otimes_R M}^E)$, is an isomorphism. Therefore the map γ_M^T is an isomorphism if and only if the map $\delta_M^{T^\vee}$ is an isomorphism.

Next we show that for all $i > 0$ we have $\mathrm{Ext}_R^i(M, T^\vee) = 0$ if and only if $\mathrm{Tor}_i^R(M, T) = 0$. By [5, Remark 1.9], we have $\mathrm{Ext}_R^i(M, T^\vee) \cong \mathrm{Tor}_i^R(M, T)^\vee$. Because the Matlis dual of a module is zero if and only if the module is zero, we conclude that $\mathrm{Ext}_R^i(M, T^\vee) = 0$ if and only if $\mathrm{Tor}_i^R(M, T) = 0$ for all $i > 0$.

Next we show that for all $i > 0$ we have $\mathrm{Ext}_R^i(\mathrm{Hom}_R(M, T^\vee), T^\vee) = 0$ if and only if $\mathrm{Ext}_R^i(T, M \otimes_R T) = 0$. Hom-tensor adjointness explains the first step in the

following sequence

$$\begin{aligned} \text{Ext}_R^i(\text{Hom}_R(M, T^\vee), T^\vee) &\cong \text{Ext}_R^i((M \otimes_R T)^\vee, T^\vee) \\ &\cong \text{Ext}_R^i(T^{\vee\vee}, (M \otimes_R T)^{\vee\vee}) \\ &\cong \text{Ext}_R^i(T, M \otimes_R T). \end{aligned}$$

The second step follows from Lemma 1.4 and the fact that T is artinian and thus Matlis reflexive. The third step follows from the fact that T and $M \otimes_R T$ are artinian and hence Matlis reflexive; see [5, Corollary 3.9]. \square

Corollary 3.8. *Assume that R is complete and let T be a quasidualizing R -module. Then $\mathcal{G}_{T^\vee}^{\text{noeth}}(R) = \mathcal{A}_T^{\text{noeth}}(R)$ and $\mathcal{G}_T^{\text{artin}}(R) = \mathcal{A}_T^{\text{artin}}(R)$.*

Proposition 3.9. *If R is complete and T is a quasidualizing R -module, then*

$$\mathcal{G}_T^{\text{mr}}(R) = \mathcal{A}_{T^\vee}^{\text{mr}}(R).$$

Proof. Let M be a Matlis reflexive R -module. We show that M satisfies the defining conditions of $\mathcal{G}_T^{\text{mr}}(R)$ if and only if M satisfies the defining conditions of $\mathcal{A}_{T^\vee}^{\text{mr}}(R)$. For the isomorphisms, consider the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\delta_M^T} & \text{Hom}_R(\text{Hom}_R(M, T), T) \\ \downarrow \gamma_M^{T^\vee} & & \downarrow \\ \text{Hom}_R(T^\vee, T^\vee \otimes_R M) & & \\ \downarrow \text{Hom}_R(T^\vee, \delta_{T^\vee \otimes M}^E) & & \\ \text{Hom}_R(T^\vee, (T^\vee \otimes_R M)^{\vee\vee}) & \xrightarrow{\text{Hom}_R(\text{Hom}_R(M, T), \delta_T^E)} \cong & \\ \downarrow \cong & & \\ \text{Hom}_R(\text{Hom}_R(T^\vee \otimes_R M, E), T^{\vee\vee}) & & \\ \downarrow \cong & & \\ \text{Hom}_R(\text{Hom}_R(M, T^{\vee\vee}), T^{\vee\vee}) & \xrightarrow[\text{Hom}_R(\text{Hom}_R(M, \delta_T^E), T^{\vee\vee})]{\cong} & \text{Hom}_R(\text{Hom}_R(M, T), T^{\vee\vee}) \end{array}$$

where the unlabeled isomorphisms are Hom-tensor adjointness and Hom-swap. Since T is artinian and hence Matlis reflexive, both the right hand map and the bottom map are isomorphisms. The module $T^\vee \otimes_R M$ is Matlis reflexive by [5, Corollary 3.6]. Thus the map $\delta_{T^\vee \otimes M}^E$ and hence the map $\text{Hom}_R(T^\vee, \delta_{T^\vee \otimes M}^E)$ is an isomorphism. Therefore the map $\gamma_M^{T^\vee}$ is an isomorphism if and only if the map δ_M^T is an isomorphism.

Next we show that for all $i > 0$ we have $\text{Ext}_R^i(M, T) = 0$ if and only if $\text{Tor}_i^R(T^\vee, M) = 0$. The fact that T is artinian and hence Matlis reflexive explains the first step in the following sequence

$$\text{Ext}_R^i(M, T) \cong \text{Ext}_R^i(M, T^{\vee\vee}) \cong \text{Tor}_i^R(M, T^\vee)^\vee \cong \text{Tor}_i^R(T^\vee, M)^\vee.$$

The second step follows from [5, Remark 1.9] and the last step follows from the commutativity of the tensor product. Because the Matlis dual of a module is zero if and only if the module is zero, we conclude that $\text{Ext}_R^i(M, T) = 0$ if and only if $\text{Tor}_i^R(T^\vee, M) = 0$ for all $i > 0$.

Next we show that for all $i > 0$ we have $\text{Ext}_R^i(\text{Hom}_R(M, T), T) = 0$ if and only if $\text{Ext}_R^i(T^\vee, T^\vee \otimes_R M) = 0$. The fact that T is artinian and hence Matlis reflexive explains the first and third steps in the following sequence

$$\begin{aligned} \text{Ext}_R^i(\text{Hom}_R(M, T), T) &\cong \text{Ext}_R^i(\text{Hom}_R(M, T^{\vee\vee}), T) \\ &\cong \text{Ext}_R^i(\text{Hom}_R(M \otimes_R T^\vee, E), T) \\ &\cong \text{Ext}_R^i(\text{Hom}_R(M \otimes_R T^\vee, E), T^{\vee\vee}) \\ &\cong \text{Ext}_R^i(T^\vee, M \otimes_R T^\vee). \end{aligned}$$

The second step follows from Hom-tensor adjointness and the last step follows from Lemma 1.4. \square

Corollary 3.10. *Assume that R is complete and let T be a quasidualizing R -module. Then $\mathcal{G}_T^{\text{noeth}}(R) = \mathcal{A}_{T^\vee}^{\text{noeth}}(R)$ and $\mathcal{G}_T^{\text{artin}}(R) = \mathcal{A}_{T^\vee}^{\text{artin}}(R)$.*

The above results show that the classes $\mathcal{G}_T^{\text{mr}}(R)$, $\mathcal{G}_T^{\text{artin}}(R)$, and $\mathcal{G}_T^{\text{noeth}}(R)$ do not exhibit some of the same properties as the class $\mathcal{G}_C^{\text{noeth}}(R)$, where C is semidualizing. For instance, we consider the following property. We say a class of R -modules \mathcal{C} satisfies the two-of-three condition if given an exact sequence of R -module homomorphisms $0 \rightarrow L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow 0$, when any two of the modules are in \mathcal{C} , so is the third. The two-of-three condition holds for some classes of modules and not for others. For example, the class of noetherian modules and the class of artinian modules both satisfy the two-of-three condition. On the other hand, the class $\mathcal{G}_C^{\text{noeth}}(R)$ does not satisfy the two-of-three condition when C is semidualizing. In contrast, the next result shows that the class $\mathcal{G}_T^{\text{full}}(R)$ satisfies the two-of-three condition when the ring is complete. This is somewhat surprising since the definitions of $\mathcal{G}_C^{\text{noeth}}(R)$ and $\mathcal{G}_T^{\text{full}}(R)$ are so similar. First we need a lemma. In the language of [3], it says that quasidualizing implies faithfully quasidualizing.

Lemma 3.11. *Let L and T be R -modules such that T is quasidualizing. If one has $\text{Hom}_R(L, T) = 0$, then $L = 0$.*

Proof. Assume that $\text{Hom}_R(L, T) = 0$.

Case 1: $T = E$. Because $\text{Hom}_R(L, E) = 0$, we have $L^{\vee\vee} = 0$. Since the map $\delta_L^E : L \rightarrow L^{\vee\vee}$ is injective by Fact 1.2, we conclude that $L = 0$.

Case 2: R is complete. Then T is Matlis reflexive and we have $0 = \text{Hom}_R(L, T) \cong \text{Hom}_R(T^\vee, L^\vee)$ from Lemma 1.4. Since T^\vee is semidualizing by Proposition 3.1, we have $L^\vee = 0$ by [3, Proposition 3.6]. By Case 1, we conclude that $L = 0$.

Case 3: the general case. The first step in the following sequence is by assumption

$$0 = \text{Hom}_R(L, T) \cong \text{Hom}_R(L, \text{Hom}_{\widehat{R}}(\widehat{R}, T)) \cong \text{Hom}_{\widehat{R}}(\widehat{R} \otimes_R L, T).$$

The second step follows from the fact that T is artinian and hence has an \widehat{R} structure and the third step is from Hom-tensor adjointness. Since T is a quasidualizing \widehat{R} -module, we can apply Case 2 to conclude that $\widehat{R} \otimes_R L = 0$. Then $L = 0$ because \widehat{R} is faithfully flat over R . \square

Question 3.12. Does a version of Lemma 3.11 hold for $T \otimes_R -$ as in [3]?

Theorem 3.13. *Assume that R is complete and let T be a quasidualizing R -module. Then $\mathcal{G}_T^{\text{full}}(R)$ satisfies the two-of-three condition.*

Proof. Let

$$(3.13.1) \quad 0 \rightarrow L_1 \xrightarrow{f} L_2 \xrightarrow{g} L_3 \rightarrow 0$$

be an exact sequence of R -module homomorphisms and let $(-)^T = \text{Hom}_R(-, T)$. There are two conditions to check and three cases. We will deal with the case when $L_1, L_2 \in \mathcal{G}_T^{full}(R)$. The case where $L_2, L_3 \in \mathcal{G}_T^{full}(R)$ is similar. The case where $L_1, L_3 \in \mathcal{G}_T^{full}(R)$ is also similar but easier.

Assume that $L_1, L_2 \in \mathcal{G}_T^{full}(R)$. Then we have $\text{Ext}_R^i(L_1, T) = 0 = \text{Ext}_R^i(L_2, T)$ for all $i > 0$. The following portion of the long exact sequence in $\text{Ext}_R^i(-, T)$ associated to the short exact sequence (3.13.1)

$$(3.13.2) \quad \cdots \rightarrow \text{Ext}_R^{i-1}(L_1, T) \rightarrow \text{Ext}_R^i(L_3, T) \rightarrow \text{Ext}_R^i(L_2, T) \rightarrow \text{Ext}_R^i(L_1, T) \rightarrow \cdots$$

shows that $\text{Ext}_R^i(L_3, T) = 0$ for all $i > 1$. For the case where $i = 1$, we apply $(-)^T$ to the following portion of the long exact sequence

$$0 \rightarrow (L_3)^T \rightarrow (L_2)^T \rightarrow (L_1)^T \rightarrow \text{Ext}_R^1(L_3, T) \rightarrow 0$$

to obtain exactness in the top row of the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\text{Ext}_R^1(L_3, T))^T & \longrightarrow & (L_1)^{TT} & \xrightarrow{f^{TT}} & (L_2)^{TT} \\ & & & & \uparrow \cong \delta_{L_1}^T & & \uparrow \cong \delta_{L_2}^T \\ & & & & 0 & \xrightarrow{f} & L_2. \end{array}$$

Since f is an injective map, the diagram shows that f^{TT} is an injective map. Hence we have $(\text{Ext}_R^1(L_3, T))^T = 0$. From Lemma 3.11 we conclude that $\text{Ext}_R^1(L_3, T) = 0$.

Next we show that $\text{Ext}_R^i(\text{Hom}_R(L_3, T), T) = 0$ for all $i > 0$. From the argument above we have the exact sequence

$$(3.13.3) \quad 0 \rightarrow (L_3)^T \rightarrow (L_2)^T \rightarrow (L_1)^T \rightarrow 0.$$

In a similar, but easier, manner than above, the long exact sequence in $\text{Ext}_R^i(-, T)$ shows that if $L_1, L_2 \in \mathcal{G}_T^{full}(R)$, then $\text{Ext}_R^i(\text{Hom}_R(L_3, T), T) = 0$ for all $i > 0$.

Lastly, we show that the map $\delta_{L_3}^T$ is an isomorphism. From the short exact sequence (3.13.1) and as a consequence of the above argument together with the short exact sequence (3.13.3), we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{f} & L_2 & \xrightarrow{g} & L_3 & \longrightarrow & 0 \\ & & \downarrow \cong \delta_{L_1}^T & & \downarrow \cong \delta_{L_2}^T & & \downarrow \delta_{L_3}^T & & \\ 0 & \longrightarrow & (L_1)^{TT} & \xrightarrow{f^{TT}} & (L_2)^{TT} & \xrightarrow{g^{TT}} & (L_3)^{TT} & \longrightarrow & 0. \end{array}$$

Since L_1, L_2 are in $\mathcal{G}_T^{full}(R)$, the maps $\delta_{L_1}^T$ and $\delta_{L_2}^T$ are isomorphisms. By the Snake Lemma, we conclude that $\delta_{L_3}^T$ is an isomorphism. \square

Corollary 3.14. *Assume that R is complete and let T be a quasidualizing R -module. Then $\mathcal{G}_T^{artin}(R) = \mathcal{A}_{T^\vee}^{artin}(R)$, $\mathcal{G}_T^{noeth}(R) = \mathcal{A}_{T^\vee}^{noeth}(R)$, and $\mathcal{G}_T^{mr}(R)$ satisfy the two-of-three condition.*

Proof. Apply Theorem 3.13 and Corollary 3.10. \square

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